Flips and variation of moduli scheme of sheaves on a surface

Yamada, Kimiko

代数幾何学シンポジウム記録
2008年度 pp.135-135

Flips and variation of moduli scheme of sheaves on a surface

Kimiko Yamada
Dept. of Math., Kyoto Univ., Japan. kyamada@math.kyoto-u.ac.jp

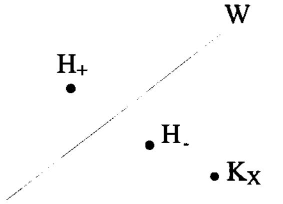
Let H be an ample line bundle on a non-singular projective surface X over $\mathbb{C}$. Denote by $M(H)$ the coarse moduli scheme of rank-two H-stable sheaves on X with Chern classes (r, c_1, c_2). We shall consider birational aspects of the problem how $\bar{M}(H)$ changes as H varies. See arXiv:0811.3522 for details.

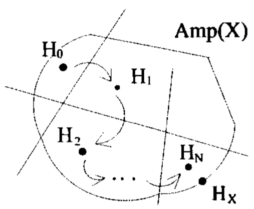
There is a union of hyperplanes $W \subset \mathrm{Amp}(X)$ called (c_1, c_2)-walls in the ample cone $\mathrm{Amp}(X)$ such that $M(H)$ changes only when H passes through walls. Let H and H_+ be ample line bundles separated by just one wall W, and $H_0 = tH \ + (1-t)H_+$ lie in W. (More exactly, we also consider parabolic stability.) For simplicity we assume that $M_\pm$ are compact, that is valid if $c_1 = 0$ and c_2 is odd for example. Denote $M_\pm = M(H_\pm)$ and $M_0 = M(H_0)$. There are natural morphisms $f \ : M \ \to M_0$ and $f_+ : M_+ \to M_0$. Let $f : X \to Y$ be a birational proper morphism such that K_X is $\mathbb{Q}$-Cartier and $-K_X$ is f-ample, and that the codimension of the exceptional set $\mathrm{Ex}(f)$ of f is more than 1. We say a birational proper morphism $f_+ : X_+ \to Y$ is a *flip* of f if (1) K_{X_+} is $\mathbb{Q}$-Cartier, (2) K_{X_+} is f_+-ample and (3) the codimension of the exceptional set $\mathrm{Ex}(f_+)$ is more than 1.

Theorem 0.1. *Assume c_2 is sufficiently large. Suppose K_X does not lies in the wall W separating H and H_+, and that K_X and H lie in the same connected components of $\mathrm{NS}(X)_{\mathbb{R}} \setminus W$. (See the left gure below.) Then the birational map*

(1)
$$\begin{tikzcd}
M_+ \arrow[rr, dotted] \arrow[dr, "f_-"'] & & M \arrow[dl, "f_+"] \\
& M_0 &
\end{tikzcd}$$

is a flip.





Suppose $M(H)$ is compact, and let us observe this theorem in case where X is minimal and $\kappa(X) \geq 1$. There is an ample line bundle H_X such that no wall of type (c_1, c_2) divides K_X and H_X. When $H \in \{(1-t)H_0 + tK_X | t \in [0,1)\}$ starts from a polarization H_0 and gets closer to K_X, one gets a finite sequence of flips

$$M(H = H_0) \cdots > M(H_1) \quad \cdots > M(H_N = H_X),$$

which terminates in $M(H_X)$. (See the right figure above.) It is known that the canonical divisor of $M(H_X)$ is nef. Thus one can regard this "natural" process described in a moduli-theoretic way as an analogy of minimal model program of $M(H)$, although it is unknown whether $M(H_X)$ admits only terminal singularities.